

Construction of orthonormal wavelet-like bases

Patricia Mariela Morillas^{a)}*Instituto de Matemática Aplicada San Luis, UNSL-CONICET, Ejército de los Andes 950, 5700 San Luis, Argentina*

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A general method for constructing wavelet-like bases in a Hilbert space \mathcal{H} starting from any orthonormal basis in \mathcal{H} and any periodic orthonormal wavelet basis is presented. With this method we can take advantage of the characteristics of both types of bases to obtain orthonormal wavelet-like bases that are suitable to represent functions and operators efficiently. © 2010 American Institute of Physics.

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I. INTRODUCTION

The applicability of wavelets in different areas of pure and applied mathematics depends on their ability to represent a wide class of functions and operators efficiently (for a treatment of the theory of wavelets see, e.g., Refs. 3, 10, 13, 14, and 16). In particular, this fact is useful in the development of fast adaptive numerical algorithms (see, e.g., Refs. 1, 2, 5, and 6). As mentioned in Ref. 15 (see also Ref. 11), wavelets are applied to the study of problems in different physical fields such as astrophysics, turbulence, meteorology, plasma physics, atomic and solid state physics, multifractals occurring in physics, biophysics (in medicine and physiology), and computational and mathematical physics.

New examples of wavelets with good properties are always required. One important property is that functions and operators should have representations in the wavelet basis with only a small number of significant coefficients, i.e., effective sparse representations. If wavelet bases are associated with a multiresolution analysis, we can change the level of the approximation of functions and operators using wavelet transforms. Smooth wavelets are important to represent differential operators. Explicit expressions of wavelets are useful for performing computations.

Based on the ideas introduced in Refs. 7 and 9 but using a different approach, we present in this paper a general method for constructing wavelet-like bases in a Hilbert space \mathcal{H} , associated with multiresolution analysis, starting from any orthonormal basis in \mathcal{H} and any periodic orthonormal wavelet basis. With this method we can take advantage of the characteristics of both types of bases to obtain orthonormal wavelet-like bases that are suitable to represent functions and operators efficiently. The method admits variants, and, in particular, we can obtain smoothness, explicit expressions, and effectively sparse representations for the wavelet-like functions.

The construction method is the combination of two procedures for obtaining orthonormal wavelet-like bases. In one of them, we use any given unitary operator and any given periodic orthonormal wavelet basis. This procedure is intended to lead to orthonormal wavelet-like bases that permit effectively sparse representations. The second procedure uses any given unitary matrix and any given orthonormal basis in \mathcal{H} and permits to obtain smoothness and explicit expressions. We then combine these two procedures for constructing smooth wavelet-like functions with explicit expressions that permit effectively sparse representations.

We illustrate the method by obtaining orthonormal wavelet-like bases based on trigonometric functions and eigenfunctions of Schrödinger and Chebyshev differential operators. We show that

^{a)}Electronic mail: morillas@unsl.edu.ar.

using eigenfunctions of an operator L , if S is certain perturbation of L , then the stiffness matrix in the wavelet-Galerkin scheme for differential equations associated with S can be preconditioned by a diagonal matrix so that the condition number is uniformly bounded.

The rest of this paper is organized as follows. In Sec. II we introduce the concepts of multi-resolution analysis, scaling-like and wavelet-like vectors in a separable Hilbert space. We also establish some basic results. In Sec. III we present the general method for constructing wavelet-like bases. Then, in Sec. IV, some examples of wavelet-like bases based on trigonometric functions and eigenfunctions of differential operators are presented. Finally, some final remarks are included in Sec. V.

II. WAVELET-LIKE ORTHONORMAL BASES IN A HILBERT SPACE

In this section we introduce some basic concepts and results that will be use in the rest of the paper. Let \mathcal{H} be a separable Hilbert space and let $j_0 \geq 0$.

Definition 2.1: The sequence $\{V_j: j \geq j_0\}$ is a multiresolution analysis (MRA) of \mathcal{H} if

- (1) V_j is a subspace of \mathcal{H} , $j \geq j_0$;
- (2) the dimension of V_j is 2^j , $j \geq j_0$;
- (3) $V_j \subset V_{j+1}$, $j \geq j_0$;
- (4) $\bigcup_{j=j_0}^{\infty} V_j$ is dense in \mathcal{H} .

Theorem 2.2: If $\{V_j: j \geq j_0\}$ is a MRA of \mathcal{H} , $W_j = V_{j+1} \cap V_j^\perp = V_{j+1} \ominus V_j$ and $j_1 \geq j_0$, then $\mathcal{H} = V_{j_1} \oplus (\bigoplus_{j=j_1}^{\infty} W_j)$.

Definition 2.3: If

- (1) $\{\Phi_{j,k}\}_{k=0}^{2^j-1}$, $j \geq j_0$, is an orthonormal system in \mathcal{H} ;
- (2) the sequence $V_j = \text{span}\{\Phi_{j,k}\}_{k=0}^{2^j-1}$, $j \geq j_0$, is a MRA of \mathcal{H} ;

then $\bigcup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is called a scaling-like system in \mathcal{H} and each $\Phi_{j,k}$ is called a scaling-like vector.

Definition 2.4: If

- (1) $\bigcup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal system in \mathcal{H} ;
- (2) there exists a scaling-like system $\bigcup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ for \mathcal{H} , such that for all $j \geq j_0$, $k, k' = 0, \dots, 2^j-1$, $\langle \Phi_{j,k}, \Psi_{j,k'} \rangle_{\mathcal{H}} = 0$;

then $\bigcup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is called a wavelet-like system for \mathcal{H} , associated with $\bigcup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$, and each $\Psi_{j,k}$ is called a wavelet-like vector.

Theorem 2.5: If $\bigcup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\bigcup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ and W_j is as in Theorem 2.2, then $W_j = \text{span}\{\Psi_{j,k}\}_{k=0}^{2^j-1}$.

From Theorem 2.2 and Theorem 2.5, we obtain the following.

Theorem 2.6: If $\bigcup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\bigcup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ and $j_1 \geq j_0$, then $\{\Phi_{j_1,k}\}_{k=0}^{2^{j_1}-1} \cup \bigcup_{j=j_1}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of \mathcal{H} .

We will call the basis of Theorem 2.6 a wavelet-like basis for \mathcal{H} .

Remark 2.7: As a consequence of the finite dimensionality, each MRA of \mathcal{H} , $\{V_j: j \geq j_0\}$, has an associated scaling-like system $\bigcup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$, such that $V_j = \text{span}\{\Phi_{j,k}\}_{k=0}^{2^j-1}$, $j \geq j_0$. Also, for each scaling-like system for \mathcal{H} , there exists an associated wavelet-like system.

In the sequel, (p_j) denotes the column vector with components $p_{j,k}$, $k=0, \dots, 2^j-1$, where each $p_{j,k}$ is an element of a certain class. For example, (Φ_j) denotes the column vector with components $\Phi_{j,k}$, $k=0, \dots, 2^j-1$, and if $f \in \mathcal{H}$, then $(\langle f, \Phi_j \rangle_{\mathcal{H}})$ denotes the column vector with components $\langle f, \Phi_{j,k} \rangle_{\mathcal{H}}$, $k=0, \dots, 2^j-1$.

Definition 2.8: Let $j \geq 1$. If H_j, G_j are matrices of order $2^{j-1} \times 2^j$, such that $H_j H_j^* = G_j G_j^* = I$, $H_j G_j^* = G_j H_j^* = O$, and $H_j^* H_j + G_j^* G_j = I$, then (H_j, G_j) will be called a filter pair.

Theorem 2.9: $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$, if and only if there exists $j_1 \geq j_0$, such that $\{\Phi_{j_1,k}\}_{k=0}^{2^{j_1}-1} \cup \cup_{j=j_1}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of \mathcal{H} and there exist (unique) filter pairs, (H_j, G_j) , $j \geq j_0 + 1$, such that for $j \geq j_0 + 1$,

$$(\Phi_{j-1}) = H_j(\Phi_j), \quad (\Psi_{j-1}) = G_j(\Phi_j), \quad (\Phi_j) = H_j^*(\Phi_{j-1}) + G_j^*(\Psi_{j-1}). \quad (1)$$

Proof: Suppose $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$. Let H_j, G_j , $j \geq j_0 + 1$, be the matrices of order $2^{j-1} \times 2^j$, with entries $\langle \Phi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}}$ and $\langle \Psi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}}$, $k=0, \dots, 2^{j-1}-1$, $k'=0, \dots, 2^j-1$, respectively. Since $V_j, W_j \subset V_{j+1}$, $V_{j+1} = V_j \oplus W_j$, $\{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of V_j and $\{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of W_j , $j \geq j_0$, it is easy to see that (1) holds and (H_j, G_j) are filter pairs. Moreover, by Theorem 2.6, for all $j_1 \geq j_0$, $\{\Phi_{j_1,k}\}_{k=0}^{2^{j_1}-1} \cup \cup_{j=j_1}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of \mathcal{H} .

Suppose now that there exists $j_1 \geq j_0$, such that $\{\Phi_{j_1,k}\}_{k=0}^{2^{j_1}-1} \cup \cup_{j=j_1}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of \mathcal{H} and there exist filter pairs, (H_j, G_j) , $j \geq j_0 + 1$, such that (1) holds. Let $V_j = \text{span}\{\Phi_{j,k}\}_{k=0}^{2^j-1}$, $W_j = \text{span}\{\Psi_{j,k}\}_{k=0}^{2^j-1}$, $j \geq j_0$. Since $\{\Phi_{j_1,k}\}_{k=0}^{2^{j_1}-1} \cup \cup_{j=j_1}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal system in \mathcal{H} and (H_j, G_j) are filter pairs, by (1), $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1} \cup \cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal system in \mathcal{H} , H_j, G_j , $j \geq j_0 + 1$, are the matrices with entries $\langle \Phi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}}$ and $\langle \Psi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}}$, $k=0, \dots, 2^{j-1}-1$, $k'=0, \dots, 2^j-1$, respectively, and $V_j, W_j \subset V_{j+1}$, $V_{j+1} = V_j \oplus W_j$. Since $\{\Phi_{j_1,k}\}_{k=0}^{2^{j_1}-1} \cup \cup_{j=j_1}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an orthonormal basis of \mathcal{H} , then $\mathcal{H} = V_{j_1} \oplus (\oplus_{j=j_1}^{\infty} W_j)$ and thus $\cup_{j=j_0}^{\infty} V_j$ is dense in \mathcal{H} . We conclude that $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} and $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an associated wavelet-like system. ■

In view of Theorem 2.9 we say that (H_j, G_j) , $j \geq j_0 + 1$, are the filter pairs of $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ and $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$. H_j and G_j are the matrices with entries $\langle \Phi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}}$ and $\langle \Psi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}}$, $k=0, \dots, 2^{j-1}-1$, $k'=0, \dots, 2^j-1$, respectively.

As a consequence of Theorem 2.9, we obtain the following.

Theorem 2.10: If $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$, filter pairs (H_j, G_j) , $j \geq j_0 + 1$, and $f \in \mathcal{H}$, then

$$(\langle f, \Phi_{j-1} \rangle_{\mathcal{H}}) = H_j(\langle f, \Phi_j \rangle_{\mathcal{H}}), \quad (\langle f, \Psi_{j-1} \rangle_{\mathcal{H}}) = G_j(\langle f, \Phi_j \rangle_{\mathcal{H}}), \quad (2)$$

$$(\langle f, \Phi_j \rangle_{\mathcal{H}}) = H_j^*(\langle f, \Phi_{j-1} \rangle_{\mathcal{H}}) + G_j^*(\langle f, \Psi_{j-1} \rangle_{\mathcal{H}}). \quad (3)$$

Although the concepts of MRA, scaling-like and wavelet-like vectors, and their properties are simple, to exhibit concrete interest examples for particular Hilbert spaces \mathcal{H} can be a difficult task. The most studied case is $\mathcal{H} = L^2(\mathbb{T})$. The simplest way to construct a MRA for $L^2(\mathbb{T})$ is the periodization (see, e.g., Refs. 10, 13, and 16). Let $\phi, \psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ be a scaling function and a wavelet, respectively, of a multiresolution analysis on the real line. Let $\phi_{j,k}^{per}, \psi_{j,k}^{per}$, $j \geq 0$, be the functions obtained by the periodization of the scaling functions $\phi_{j,k}$ and the wavelets $\psi_{j,k}$, respectively, i.e.,

$$\phi_{j,k}^{per} = \sum_{l \in \mathbb{Z}} \phi_{j,k}(x+l) = 2^{j/2} \sum_{l \in \mathbb{Z}} \phi(2^j(x+l) - k) \quad (4)$$

and

$$\psi_{j,k}^{per} = \sum_{l \in \mathbb{Z}} \psi_{j,k}(x+l) = 2^{j/2} \sum_{l \in \mathbb{Z}} \psi(2^j(x+l) - k). \quad (5)$$

In the sequel we suppose that both ϕ, ψ have radial decreasing $L^1(\mathbb{R})$ -majorants [for a definition of radial decreasing $L^1(\mathbb{R})$ -majorant see (3.1) of Chap. 5 in Ref. 13]. In this case $\cup_{j=j_0}^{\infty} \{\phi_{j,k}^{per}\}_{k=0}^{2^j-1}$ is a

scaling-like system for $L^2(\mathbb{T})$ with an associated wavelet-like system $\cup_{j=j_0}^{\infty} \{\psi_{j,k}^{per}\}_{k=0}^{2^j-1}$ (see, e.g., Ref. 13).

Lemma 2.11: Suppose that $\phi_{j,k} = \sum_{m \in \mathbb{Z}} a_{k,m}^{j+1} \phi_{j+1,m}$ and $\psi_{j,k} = \sum_{m \in \mathbb{Z}} b_{k,m}^{j+1} \phi_{j+1,m}$. If H_j^{per}, G_j^{per} are the matrices of order $2^{j-1} \times 2^j$, $j > 0$, with entries $H_j^{per}(k,m) = \sum_{s \in \mathbb{Z}} a_{k,m+2^j s}^j$ and $G_j^{per}(k,m) = \sum_{s \in \mathbb{Z}} b_{k,m+2^j s}^j$, then

$$(\phi_{j-1}^{per}) = H_j^{per}(\phi_j^{per}), \quad (6)$$

$$(\psi_{j-1}^{per}) = G_j^{per}(\phi_j^{per}), (\phi_j^{per}) = (H_j^{per})^*(\phi_{j-1}^{per}) + (G_j^{per})^*(\psi_{j-1}^{per}). \quad (7)$$

As a consequence of the above lemma, (H_j^{per}, G_j^{per}) , $j > 0$, are the filter pairs of $\cup_{j=j_0}^{\infty} \{\phi_{j,k}^{per}\}_{k=0}^{2^j-1}$ and $\cup_{j=j_0}^{\infty} \{\psi_{j,k}^{per}\}_{k=0}^{2^j-1}$.

III. CONSTRUCTION OF WAVELET-LIKE ORTHONORMAL BASES

In this section we describe procedures to obtain a MRA of \mathcal{H} .

A. Construction of a MRA of \mathcal{H} starting from a MRA of $L^2(\mathbb{T})$ and a unitary operator $U: L^2(\mathbb{T}) \rightarrow \mathcal{H}$

Theorem 3.1: $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ and filter pairs $(H_j, G_j) = (H_j^{per}, G_j^{per})$ if and only if there exists a (unique) unitary operator $U: L^2(\mathbb{T}) \rightarrow \mathcal{H}$, such that

$$\Phi_{j,k} = U \phi_{j,k}^{per} \quad (8)$$

and

$$\Psi_{j,k} = U \psi_{j,k}^{per}. \quad (9)$$

Proof: Suppose that $U: L^2(\mathbb{T}) \rightarrow \mathcal{H}$ is a unitary operator such that $\Phi_{j,k} = U \phi_{j,k}^{per}$ and $\Psi_{j,k} = U \psi_{j,k}^{per}$.

Since U preserves inner products, $\{\phi_{j,k}^{per}\}_{k=0}^{2^j-1}$, $j \geq j_0$, and $\cup_{j=j_0}^{\infty} \{\psi_{j,k}^{per}\}_{k=0}^{2^j-1}$ are orthonormal systems in $L^2(\mathbb{T})$, then $\{\Phi_{j,k}\}_{k=0}^{2^j-1}$, $j \geq j_0$, and $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ are orthonormal systems in \mathcal{H} . Therefore, $\{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is linearly independent and thus the dimension of V_j is 2^j .

We also have $V_j = UV_j^{per}$ and $W_j = UW_j^{per}$. Then $V_j = UV_j^{per} \subset UV_{j+1}^{per} = V_{j+1}$.

Let $g \in \mathcal{H}$ and $\epsilon > 0$. Since U is onto, there exists $f \in L^2(\mathbb{T})$, such that $g = Uf$. Since $\cup_{j=j_0}^{\infty} V_j^{per}$ is dense in $L^2(\mathbb{T})$, there exists $f_{\epsilon} \in \cup_{j=j_0}^{\infty} V_j^{per}$, such that $\|f - f_{\epsilon}\|_{L^2(\mathbb{T})} < \epsilon$. Suppose $f_{\epsilon} \in V_j^{per}$. Then $Uf_{\epsilon} \in V_j$ and $\|g - Uf_{\epsilon}\|_{\mathcal{H}} = \|Uf - Uf_{\epsilon}\|_{\mathcal{H}} = \|f - f_{\epsilon}\|_{L^2(\mathbb{T})} < \epsilon$. This shows that $\cup_{j=j_0}^{\infty} V_j$ is dense in \mathcal{H} .

Thus we have proven that $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} .

Since U preserves inner products, we also have for $j \geq j_0$, $k, k' = 0, \dots, 2^j - 1$, $\langle \Phi_{j,k}, \Psi_{j,k'} \rangle_{\mathcal{H}} = \langle \phi_{j,k}^{per}, \psi_{j,k'}^{per} \rangle_{L^2(\mathbb{T})} = 0$. Then $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ is an associated wavelet-like system.

Finally, for $k = 0, \dots, 2^{j-1} - 1$ and $k' = 0, \dots, 2^j - 1$, $\langle \Phi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}} = \langle \phi_{j-1,k}^{per}, \phi_{j,k'}^{per} \rangle_{L^2(\mathbb{T})}$ and $\langle \Psi_{j-1,k}, \Phi_{j,k'} \rangle_{\mathcal{H}} = \langle \psi_{j-1,k}^{per}, \phi_{j,k'}^{per} \rangle_{L^2(\mathbb{T})}$. Then, $H_j = H_j^{per}$ and $G_j = G_j^{per}$.

Suppose now that $\cup_{j=j_0}^{\infty} \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\cup_{j=j_0}^{\infty} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ and filter pairs $(H_j, G_j) = (H_j^{per}, G_j^{per})$. For $j_1 \geq j_0$ consider the unique unitary operator $U: L^2(\mathbb{T}) \rightarrow \mathcal{H}$, such that $\Phi_{j_1,k} = U \phi_{j_1,k}^{per}$ and $\Psi_{j,k} = U \psi_{j,k}^{per}$, $j \geq j_1$. We have

$$\Phi_{j_1-1,k} = \sum_{l=0}^{2^{j_1-1}-1} H_{j_1}(k,l) \Phi_{j_1,l} = \sum_{l=0}^{2^{j_1-1}-1} H_{j_1}^{per}(k,l) U \phi_{j_1,l}^{per} = U \phi_{j_1-1,k}^{per}.$$

Following in this manner it can be proven that for $j < j_1$, $\Phi_{j,k} = U \phi_{j,k}^{per}$. Similarly, $\Psi_{j,k} = U \psi_{j,k}^{per}$ for $j < j_1$. We now have

$$\begin{aligned} \Phi_{j_1+1,k} &= \sum_{l=0}^{2^{j_1+1}-1} H_{j_1+1}^*(k,l) \Phi_{j_1,l} + \sum_{l=0}^{2^{j_1+1}-1} G_{j_1+1}^*(k,l) \Psi_{j_1,l} = \sum_{l=0}^{2^{j_1+1}-1} (H_{j_1+1}^{per})^*(k,l) U \phi_{j_1,l}^{per} \\ &+ \sum_{l=0}^{2^{j_1+1}-1} (G_{j_1+1}^{per})^*(k,l) U \psi_{j_1,l}^{per} = U \phi_{j_1+1,k}^{per}. \end{aligned} \quad (10)$$

Repeating this reasoning it can be proven that for $j > j_1$, $\Phi_{j,k} = U \phi_{j,k}^{per}$. ■

Taking into account the previous theorem, we can consider unitary operators $U: L^2(\mathbb{T}) \rightarrow \mathcal{H}$ to construct MRAs in the separable Hilbert \mathcal{H} . Given U we define $\Phi_{j,k}$ and $\Psi_{j,k}$ by (8) and (9), respectively.

Remark 3.2: If several coefficients of an element of \mathcal{H} in a wavelet-like basis are small, i.e., we have an effectively sparse representation, we say that the basis compresses these coefficients. Given $f \in \mathcal{H}$, since $\langle f, \Phi_{j,k} \rangle_{\mathcal{H}} = \langle U^* f, \phi_{j,k}^{per} \rangle_{L^2(\mathbb{T})}$ and $\langle f, \Psi_{j,k} \rangle_{\mathcal{H}} = \langle U^* f, \psi_{j,k}^{per} \rangle_{L^2(\mathbb{T})}$, then the compression of the coefficients of f depends on the compression of the coefficients of $U^* f$.

Remark 3.3: For $f \in \mathcal{H}$, let $P_n(f)$ be the projection of f onto V_n . We have

$$P_n(f) = \sum_{k=0}^{2^{j_0}-1} \langle f, \Phi_{j_0,k} \rangle_{\mathcal{H}} \Phi_{j_0,k} + \sum_{j=j_0}^n \sum_{k=0}^{2^j-1} \langle f, \Psi_{j,k} \rangle_{\mathcal{H}} \Psi_{j,k}.$$

Since U is unitary,

$$\|f - P_n(f)\|_{\mathcal{H}}^2 = \|U^* f - U^* P_n(f)\|_{L^2(\mathbb{T})}^2.$$

Thus, the rate of approximation of $P_n(f)$ to f in the norm of \mathcal{H} is the same as the rate of approximation of $U^* P_n f$ to $U^* f$ in the norm of $L^2(\mathbb{T})$. Note that

$$U^* P_n(f) = \sum_{k=0}^{2^{j_0}-1} \langle U^* f, \phi_{j_0,k}^{per} \rangle_{L^2(\mathbb{T})} \phi_{j_0,k}^{per} + \sum_{j=j_0}^n \sum_{k=0}^{2^j-1} \langle U^* f, \psi_{j,k}^{per} \rangle_{L^2(\mathbb{T})} \psi_{j,k}^{per}$$

is the projection of $U^* f$ onto V_n^{per} .

B. Construction of a MRA for \mathcal{H} starting from an orthonormal basis of \mathcal{H} , filter pairs, and a unitary matrix

Theorem 3.4: Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis of \mathcal{H} and (H_j, G_j) , $j \geq j_0 + 1$ be filter pairs. Then $\cup_{j=j_0}^\infty \{\Phi_{j,k}\}_{k=0}^{2^j-1}$ is a scaling-like system for \mathcal{H} with a wavelet-like system $\cup_{j=j_0}^\infty \{\Psi_{j,k}\}_{k=0}^{2^j-1}$ and filter pairs (H_j, G_j) , $j \geq j_0 + 1$, if and only if there exist $j_1 \geq j_0$ and a unitary matrix

$$T = \begin{pmatrix} A_{j_1} \\ B_{j_1} \\ B_{j_1+1} \\ \vdots \end{pmatrix},$$

i.e., its rows and columns form orthonormal systems in $\ell^2(\mathbb{Z})$, such that

$$(\Phi_{j_1}) := A_{j_1}(u), \quad (11)$$

$$(\Psi_j) := B_j(u), \quad j \geq j_1, \quad (12)$$

$$(\Phi_{j-1}) = H_j(\Phi_j), \quad j_0 + 1 \leq j \leq j_1, \quad (13)$$

$$(\Psi_{j-1}) = G_j(\Phi_j), \quad j_0 + 1 \leq j \leq j_1, \quad (14)$$

and

$$(\Phi_j) = H_j^*(\Phi_{j-1}) + G_j^*(\Psi_{j-1}), \quad j > j_1. \quad (15)$$

Proof: With $\bigcup_{j=j_0}^{\infty} V_j$ the matrix with entries $\langle \Phi_{j_1,k}, e_l \rangle_{\mathcal{H}}$, $k=0, \dots, 2^{j_1}-1$, $l \geq 0$ and B_j the matrix with entries $\langle \Psi_{j,k}, e_l \rangle_{\mathcal{H}}$, $k=0, \dots, 2^j-1$, $l \geq 0$, $j \geq j_1$, the proof follows immediately from Theorem 2.9. ■

Remark 3.5: If

$$M_{j,j'} = \left(\prod_{s=j'+1}^{j'} H_s \right) A_{j'}, \quad j \leq j', \quad (16)$$

$$N_{j,j'} = \left(\prod_{s=j'+1}^{j'} G_s \right) A_{j'}, \quad j \leq j', \quad (17)$$

$$M_{j,j'} = \left(\prod_{s=j'+1}^j H_{j+j'+1-s}^* \right) A_{j'}, \quad j \geq j', \quad (18)$$

and

$$N_{j,j'} = \left(\prod_{s=j'+2}^j H_{j+j'+2-s}^* \right) G_{j'+1}^* B_{j'}, \quad j > j', \quad (19)$$

then

$$(\Phi_j) := M_{j,j_1}(u), \quad j_0 \leq j \leq j_1, \quad (20)$$

$$(\Psi_j) := N_{j,j_1}(u), \quad j_0 \leq j < j_1, \quad (21)$$

$$(\Phi_j) := M_{j,j_1}(u) + \sum_{j'=j_1}^{j-1} N_{j,j'}(u), \quad j > j_1, \quad (22)$$

and

$$(\Psi_j) := B_j(u), \quad j \geq j_1. \quad (23)$$

C. A hybrid procedure

We have presented above two ways of defining a MRA on \mathcal{H} .

- (1) Take a unitary operator $U: L^2(\mathbb{T}) \rightarrow \mathcal{H}$ and define the scaling-like vectors $\Phi_{j,k}$ and wavelet-like vectors $\Psi_{j,k}$ in H as the image under U of the scaling functions $\phi_{j,k}^{per}$ and wavelets $\psi_{j,k}^{per}$, respectively. This method is described in Theorem 3.1. Moreover, by this theorem, this method is the only way to obtain a MRA in \mathcal{H} with filters H_j^{per} and G_j^{per} .

- (2) Take an orthonormal basis $\{e_n\}$ of \mathcal{H} , filters H_j, G_j and a unitary matrix T and define the scaling-like vectors $\Phi_{j,k}$ and wavelet-like vectors $\Psi_{j,k}$ in \mathcal{H} using relations (11)–(15). This method is described in Theorem 3.4.

In the first approach we work with known filters that have good properties. For example, the compression action of any wavelet (or wavelet-like) basis depends on the filters (see, e.g., Ref. 12). Moreover, as was noted in Remarks 3.2 and 3.3, the obtained MRA in \mathcal{H} and the periodic MRA in $L^2(\mathbb{T})$ have the same rate of convergence of the projections and the same compression properties.

With the second approach we can obtain scaling-like and wavelet-like vectors in \mathcal{H} that special properties. If each row of T has only a finite number of non-null entries, i.e., $\Phi_{j,k}$ and $\Psi_{j,k}$ are finite linear combinations of the elements of the basis $\{e_n\}_{n=0}^\infty$ and each e_n is, for example, C^∞ or a polynomial, $\Phi_{j,k}$ and $\Psi_{j,k}$ will also be C^∞ or a polynomial. Moreover, if each e_n has an explicit expression $\Phi_{j,k}$ and $\Psi_{j,k}$ will also have an explicit expression.

For taking the advantages of both approaches, we can combine them considering in the second procedure the filters H_j^{per}, G_j^{per} and a unitary matrix T with only a finite number of non-null entries in each row. Given $0 \leq j_0 < j_1$, a concrete and simple case is to consider $V_j = \text{span}\{e_n\}_{n=0}^{2^j-1}$ and $W_j = \text{span}\{e_n\}_{n=2^j}^{2^{j+1}-1}$ for $j \geq j_1$. Now, given the filters (H_j^{per}, G_j^{per}) , we define $(\Phi_{j-1}), (\Psi_{j-1})$, $j_0+1 \leq j \leq j_1$, using (13) and (14), respectively. The projection onto V_{j_1} of a function can have a dense representation in the original basis $\{e_n\}_{n=0}^{2^{j_1}-1}$ but can be effectively sparse in the new basis $\{\Phi_{j_0,k}\}_{k=0}^{2^{j_0}-1} \cup \bigcup_{j=j_0+1}^{j_1-1} \{\Psi_{j,k}\}_{k=0}^{2^j-1}$. In Sec. IV we present wavelet-like bases constructed in this manner. It is important to note that the coefficients corresponding to the new basis can be obtained from those corresponding to the original basis using (2).

IV. SOME EXAMPLES

In the following examples the filters (H_j^{per}, G_j^{per}) correspond to the periodic Daubechies scaling and wavelet functions with 6 vanishing moments.

A. Orthonormal trigonometric wavelet-like bases

Given $0 \leq j_0 \leq j_1$, for $j \geq j_1$, we consider in $L^2(\mathbb{T})$ the spaces

$$V_{j+1} = \text{span}\{1, \cos x, \dots, \cos(2^j - 1)x, \sin x, \dots, \sin 2^j x\}$$

and

$$W_{j+1} = \text{span}\{\cos 2^j x, \dots, \cos(2^{j+1} - 1)x, \sin(2^j + 1)x, \dots, \sin 2^{j+1} x\}.$$

These spaces are considered in Ref. 4. Considering $j_0=2$ and $j_1=7$, Fig. 1 shows the graphs of some of these scaling-like and wavelet-like functions. Figure 2 shows the pattern and the number nz of coefficients in the representation of some functions in this basis that are greater than 10^{-7} in absolute value. Considering also the number nz_o of coefficients in the representation of the functions in the original basis that are greater than 10^{-7} in absolute value, the compression ratios $c = nz_o/256$ and $cn = nz/256$ are also included. For the last function the roles of H_j and G_j were interchanged to obtain a greater compression. In all cases the coefficients in the original basis were sorted in ascending order to improve the compression action of H_j and G_j .

B. Orthonormal wavelet-like bases based on eigenfunctions

Let μ be a (Borel) measure on the interval J and consider the Hilbert space $\mathcal{H} = L^2(\mu; J)$. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis of $L^2(\mu; J)$. Let L be a self-adjoint positive differential operator defined on the set $C^\infty(J)$. Suppose that the elements of the basis $\{e_n\}_{n=0}^\infty \subset C^\infty(J)$ are eigenfunctions of L with positive eigenvalues λ_n , i.e., $Le_n = \lambda_n e_n$. Suppose now that S is a self-adjoint positive differential operator, such that $\langle Sg, g \rangle_{L^2(\mu; J)} \approx \langle Lg, g \rangle_{L^2(\mu; J)}$, $g \in C^\infty(J)$. Consider the equation

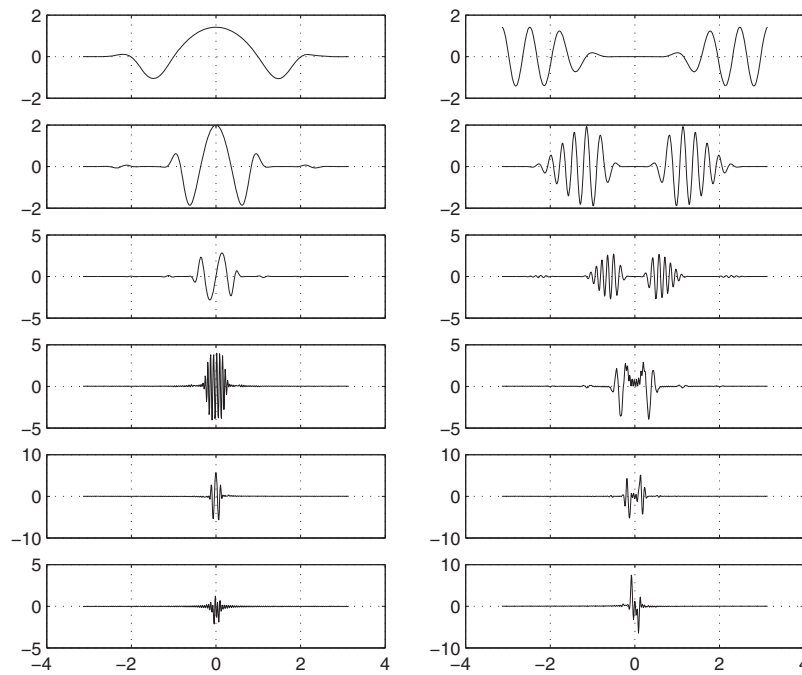


FIG. 1. Orthonormal trigonometric scaling-like functions (left column) and wavelet-like functions (right column) for $j = 7, \dots, 2$ with $k=1, 1, 17, 13, 1, 2$, respectively.

$$Su = f, \quad (24)$$

with $f \in L^2(\mu; J)$. A Galerkin scheme for Eq. (24) is to consider some $j_1 > j_0 \geq 0$ and to find $v \in V_{j_1}$, such that

$$\langle Sv, w \rangle_{L^2(\mu; J)} = \langle f, w \rangle_{L^2(\mu; J)}, \quad \forall w \in V_{j_1}. \quad (25)$$

The Galerkin scheme converges in the norm $\|g\|_L = \langle Lg, g \rangle_{L^2(\mu; J)}$. Noting that $V_{j_1} = \text{span}\{e_n\}_{n=0}^{2^{j_1}-1}$, solving (25) is equivalent to finding a 2^{j_1} -dimensional vector d , such that

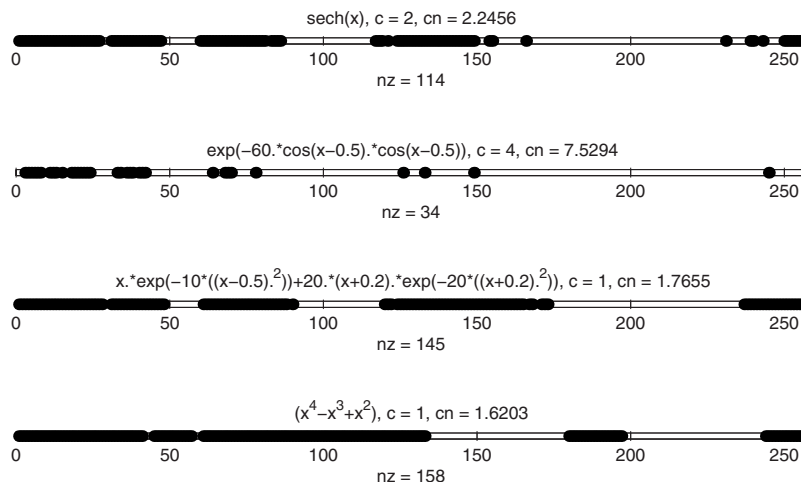


FIG. 2. Pattern and number nz of coefficients in the representation of some functions in the orthonormal trigonometric wavelet-like bases that are greater than 10^{-7} in absolute value, along with the compression coefficients c and cn . For the last function the roles of H_j and G_j were interchanged to improve the compression.

$$S_{2^{j_1}} d = s_{2^{j_1}}, \quad (26)$$

where $S_{2^{j_1}}$ is the matrix with entries $\langle S e_m, e_n \rangle_{L^2(\mu; J)}$ and $s_{2^{j_1}}$ is the vector with entries $\langle f, e_n \rangle_{L^2(\mu; J)}$. Consider the diagonal matrix $D_{2^{j_1}}$ with diagonal entries equal to $\lambda_m^{1/2}$, $m=0, \dots, 2^{j_1}-1$. (26) can be rewritten as

$$M_{2^{j_1}} x = b_{2^{j_1}}, \quad (27)$$

where $M_{2^{j_1}} = D_{2^{j_1}}^{-1} S_{2^{j_1}} D_{2^{j_1}}^{-1}$, $x = D_{2^{j_1}} d$, and $b_{2^{j_1}} = D_{2^{j_1}}^{-1} s_{2^{j_1}}$. Let $g = \sum_{n=0}^{2^{j_1}-1} \lambda_n^{-1/2} x_n e_n$. Since $\langle M_{2^{j_1}} x, x \rangle = \langle S g, g \rangle_{L^2(\mu; J)} \approx \langle L g, g \rangle_{L^2(\mu; J)} = \|x\|^2$, then the ratios of the largest and smallest eigenvalues of the matrices $M_{2^{j_1}}$ are uniformly bounded, i.e., are independent of j_1 . So, the condition numbers of the matrices $M_{2^{j_1}}$ are uniformly bounded. Since the transformation from the basis $\{e_n\}_{n=0}^{2^{j_1}-1}$ to the new basis $\{\Phi_{j_0, k}\}_{k=0}^{2^{j_0}-1} \cup \bigcup_{j=j_0}^{j_1-1} \{\Psi_{j, k}\}_{k=0}^{2^j-1}$ is orthogonal, in this new basis the conditions numbers are uniformly bounded too.

Next we apply the previous analysis to perturbed Schrödinger and Chebyshev differential equations. These equations were considered in Refs. 8 and 9, respectively (see Remark 4.3 below).

Example 4.1: Let $J = (-\infty, \infty)$, $\mu \equiv 1$, and e_n be the Hermite functions, i.e., $e_n(x) = h_n(x) e^{-x^2/2} / \sqrt{2^n n! \sqrt{\pi}}$, where h_n are the Hermite polynomials. These e_n are eigenfunctions of the Schrödinger operator $-d^2/dx + x^2 I$. Let $a(x), b(x)$ be measurable functions, such that $0 < a_1 \leq a(x) \leq a_2$, $0 < b_1/x^2 \leq b(x) \leq b_2/x^2$ for some positive constants a_1, a_2, b_1, b_2 . Then, by the previous discussion, the resultant algebraic system for the Galerkin scheme of the perturbed Schrödinger equation,

$$-(a(x)u'(x))' + b(x)u(x) = f(x), \quad (28)$$

can be preconditioned by a diagonal matrix so that the condition number is uniformly bounded. Equation (28) can be viewed as a model of second-order elliptic operators with unbounded coefficients.

Example 4.2: A similar result can be obtained for the degenerate elliptic equation,

$$-(1-x^2)^{1/2} \frac{d}{dx} \left(a(x) \frac{du}{dx} \right) = f(x), \quad (29)$$

with $0 \leq a_1(1-x^2)^{1/2} \leq a(x) \leq a_2(1-x^2)^{1/2}$, if we consider $J = [-1, 1]$, $\mu = (1-x^2)^{-1/2} dx$, $e_0(x) = 1/\sqrt{\pi}$, $e_n(x) = \sqrt{2/\pi} T_n(x)$, $n=1, 2, \dots$, where $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials. The e_n are eigenfunctions of the Chebyshev operator $-(1-x^2)^{1/2} d/dx((1-x^2)^{1/2} d/dx)$.

The graphs of some of the scaling-like and wavelet-like functions considered in Example 4.1 with $j_0=2$ and $j_1=6$, and in Example 4.2 with $j_0=2$ and $j_1=8$, appear in Figs. 3 and 4, respectively.

Remark 4.3: The eigenfunctions in the previous examples are considered in the articles^{8,9} for constructing orthonormal wavelet-like bases using a different approach. Nevertheless, if we consider the Meyer scaling function ϕ and wavelet ψ , and $j_1 \geq j_0=0$, these scaling-like and wavelet-like functions can be obtained as in Theorem 3.4 with the filter pairs of $\phi_{j,k}^{per}$, $\psi_{j,k}^{per}$, and $T(k, \sigma(n)) = \widehat{\phi_{j_1,k}}(2\pi n)$ for $k=0, \dots, 2^{j_1}-1$, $|n| < \frac{2}{3}2^{j_1}$, $T(2^j+k, \sigma(n)) = \widehat{\psi_{j,k}}(2\pi n)$ for $j \geq j_1$, $k=0, \dots, 2^j-1$, $\frac{1}{3}2^j < |n| < \frac{4}{3}2^j$, where $\sigma: \mathbb{Z} \rightarrow \mathbb{Z}^+$ is given by $\sigma(n) = 2n$ if $n \geq 0$ and $\sigma(n) = 2|n|-1$ if $n < 0$. The resultant algebraic systems for Eqs. (28) and (29) corresponding to the Galerkin scheme in these wavelet-like bases are also considered. It is proved that they can be preconditioned by a diagonal matrix and that the scheme converges in the $L^2(\mu; J)$ norm. A periodic pseudodifferential operator connected with the equation $Su = f$ is introduced.

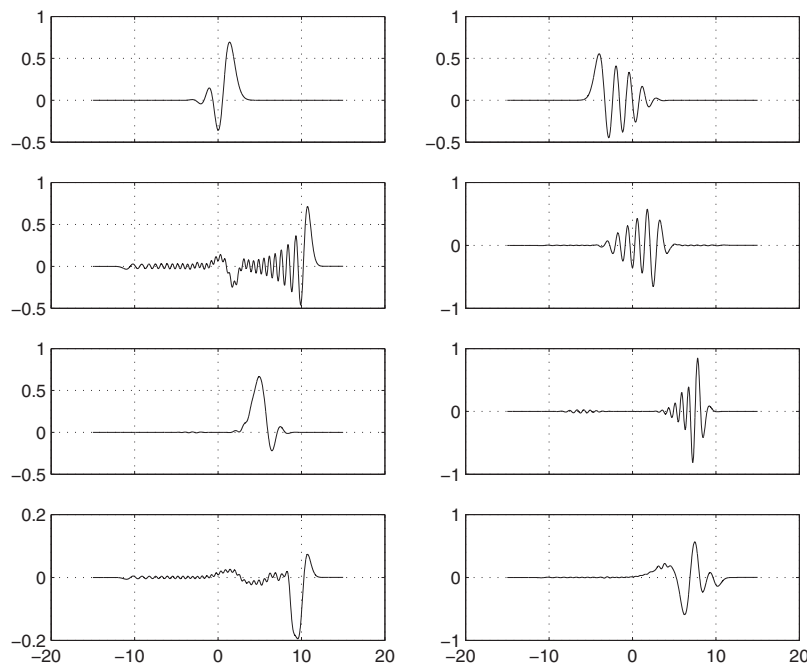


FIG. 3. Orthonormal wavelet-like bases based on Hermite polynomials. Scaling-like functions (left column) and wavelet-like functions (right column) for $j=5, 4, 3, 2$ with $k=1, 15, 1, 1$, respectively.

V. FINAL REMARKS

We have presented a general method for constructing wavelet-like bases in a Hilbert space that can lead to smooth wavelet-like functions with explicit expressions that permit effectively

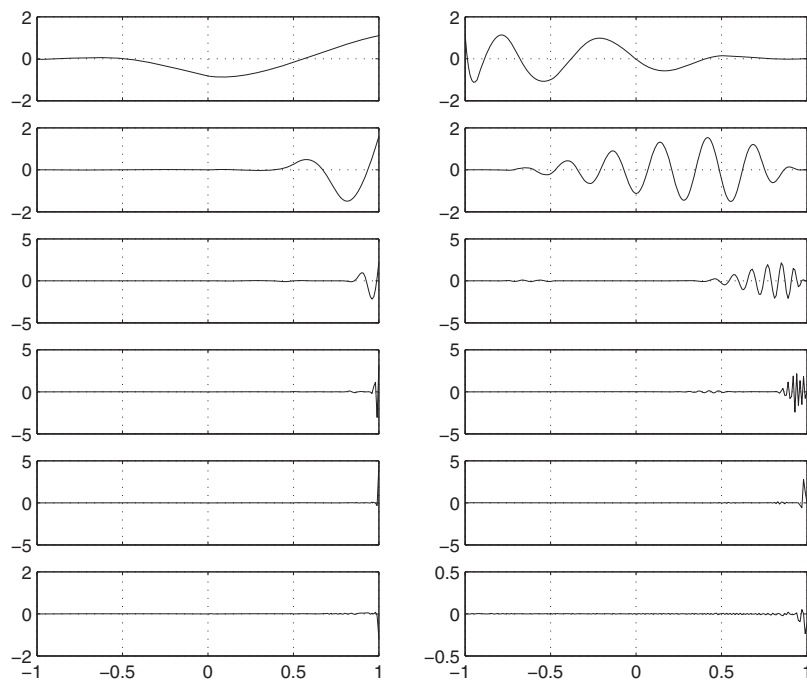


FIG. 4. Orthonormal wavelet-like bases based on Chebyshev polynomials. Scaling-like functions (left column) and wavelet-like functions (right column) for $j=7, \dots, 2$ with $k=1$, respectively.

sparse representations. The method was illustrated in a particular simple case using trigonometric functions and eigenfunctions of differential operators. Future investigations can explore other implementations of the method and other special functions. In particular, if the e_n are multivariate functions or are defined on certain domain, then we can obtain wavelet-like functions with this characteristic. Also, different filter pairs and matrices T can be used.

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